JOURNAL OF APPROXIMATION THEORY **81**, 425–428 (1995)

Note

A Note on Metric Projections*

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Communicated by Allan Pinkus

Received December 10, 1993; accepted in revised form April 4, 1994

In this note we give an example of a strictly convex, reflexive, smooth Banach space which has a Chebyshev subspace M, such that the projection onto M is linear and has norm equal to 2. Moreover, we give necessary and sufficient conditions on a space so that every projection has norm less than a constant which is less than 2. © 1995 Academic Press, Inc.

1. INTRODUCTION AND MAIN RESULTS

Let X be a normed linear space and M a subspace of X. We define $P_M(x) := \{y \in M: \|x-y\| = d(x, M)\}$ and $\|P_M\| := \sup\{\|y\|: y \in P_M(x), \|x\| \leqslant 1\}$. We trivially have $\|P_M\| \leqslant 2$ and it is well known that X is a pre-Hilbert space iff $\|P_M\| \leqslant 1$ for every subspace M of X. In this paper, we give necessary and sufficient conditions for the existence of a constant $\lambda < 2$ such that $\|P_M\| \leqslant \lambda$ for every subspace M.

In [2] Deutsch and Lambert gave an example of a Chebyshev subspace M of $C[0, 1]$ such that P_M is linear and $\|P_M\| = 2$. In Section 2 we obtain a projection with the same properties. However, the space is a strictly convex, reflexive, smooth Banach space.

A *uniformly non-square space* (UNS) is a Banach space X such that there exists a constant α, $0 < \alpha < 1$, satisfying $\|x-y\| \leqslant 2\alpha$ or $\|x+y\| \leqslant 2\alpha$ for every $x, y \in U(x)$ (where $U(X)$ is the unit ball). A *locally uniformly rotund space* (LUR) is a Banach space X such that if $x, x_n \in U(X)$ and $\|x + x_n\| \to 2$, then $\|x - x_n\| \to 0$.

* This work was partially supported by CONICOR and Universidad Nacional de Río Cuarto.

0021-9045/95 $6.00

In [4] James proved that every UNS space is reflexive. The converse is false (see the example in Section 2). If X is LUR then every Chebyshev subspace M of X is an approximatively compact set. For other properties of this space, see [1] and [4].

As usual, for $1 \leqslant p \leqslant \infty$ we let $l_p(2)$ denote the linear space $\mathbb{R}^2$ with the l_p-norm.

THEOREM 1.1. *X is a UNS space iff there exists a constant $\lambda < 2$ such that $\|P_M\| \leqslant \lambda$ for any subspace M of X.*

Proof. Assume X is a UNS space, $x \in U(X)$ and $y \in P_M(x)$. As $x - y \in U(X)$ we have $\|x-(x-y)\| \leqslant 2\alpha$ or $\|x+(x-y)\| \leqslant 2\alpha$. Thus $\|y\| \leqslant 2\alpha$ or $\|y\| \leqslant \alpha + 1$, and therefore $\|P_M\| \leqslant \alpha + 1 < 2$.

Now suppose that X is not a UNS space. Then there exist $x_n, y_n \in U(X)$ such that $\|x_n \mp y_n\| \to 2$. Let e_i, $i = 1, 2$, be the canonical vectors of $\mathbb{R}^2$, and f_n the linear functional of $l_1(2)$ into X defined by $f_n(e_1) = x_n$ and $f_n(e_2) = y_n$. Using the inequality $\|u+v\| - 1 \leqslant \|tu + (1-t)v\|$, $u, v \in U(X)$, $0 \leqslant t \leqslant 1$, we obtain

$$\|(x,y)\|_1 \min\{\|x_n \mp y_n\| - 1\} \leqslant \|f_n(x,y)\| \leqslant \|(x,y)\|_1. \tag{1}$$

Hence $|(x,y)|_n := \|f_n(x,y)\|$ defines a norm on $\mathbb{R}^2$ and $|(x,y)|_n \to \|(x,y)\|_1$. If r_n is a sequence of real positive numbers with $r_n \uparrow 2$, as a consequence of [2, Proposition 4.2] we get a sequence M_n of Chebyshev subspaces of $l_1(2)$ such that $\|P_{M_n}\|_1 = r_n$. Since f_n preserves the norm, it follows from (1), the convergence of the norms $|\cdot|_n$ to the norm $\|\cdot\|_1$, and Kripke's algorithm [3, p. 118] that there exists a sequence K_n of one dimensional subspaces of X with $\|P_{K_n}\| \geqslant n/(n+1)\, r_n$, which is a contradiction. ∎

The following theorem gives sufficient conditions on M so that $\|P_M\| < 2$. It generalizes [2, Lemma 4.3].

THEOREM 1.2. *Let M be an approximatively compact Chebyshev subspace of a normed linear space X with* codim $M < \infty$. *Then $\|P_M\| < 2$.*

Proof. Since M is an approximatively compact Chebyshev subspace, P_M is a continuous function and $\operatorname{codim} M < \infty$ implies that $P_M^{-1}(0)$ is boundedly compact, see [3, p. 168]. Suppose that there exist x_n, $\|x_n\| = 1$, with $\|P_M(x_n)\| \to 2$. Since $x_n - P_M(x_n) \in P_M^{-1}(0) \cap U(X)$ we get a subsequence n_k such that $x_{n_k} - P(x_{n_k}) \to x \in P_M^{-1}(0)$. Clearly $\|x\| = 1$, hence $d(x, M) = 1$. On the other hand $1 \leqslant \|x + P_M(x_{n_k})\| \leqslant \|x + P_M(x_{n_k}) - x_{n_k}\| + \|x_{n_k}\| \to 1$. Therefore $(-P_M(x_{n_k}))$ is a minimizing sequence of x. Since the subspace M is approximatively compact there exists a subsequence of $(-P_M(x_{n_k}))$ which converges to $P_M(x)=0$, but we have supposed that $\|P_M(x_{n_k})\| \to 2$ as $k \to \infty$. ∎

Remark. The condition $\operatorname{codim} M < \infty$ cannot be removed in the theorem as we shall show in Section 2.

2. An Example

In this section we present an example of a reflexive, strictly convex, smooth Banach space with a Chebyshev subspace whose metric projection is linear with norm equal to 2. We need to prove the following lemma.

Lemma 2.1. *Let p, $1 < p < \infty$, $q = p/(p-1)$, and H the hyperplane in $\mathbb{R}^2$ given by $\{(x, y): \alpha x + \beta y = 0\}$, where $\|(\alpha, \beta)\|_q = 1$. Then*

$$\|P_H\| = \|(\alpha, \beta)\|_p \, \|(|\alpha|^{q-1}, |\beta|^{q-1})\|_q.$$

Proof. We use the following standard fact. If f is a linear functional with $\|f\| = 1$ then $M := \operatorname{Ker} f$ is Chebyshev iff there exists a unique e with $f(e) = \|e\| = 1$ and the metric projection onto M is $P_M(x) = x - f(x)\, e$. In our case, we have

$$\begin{aligned} \|P_H(x, y)\|_p &= \|(x, y) - (\alpha x + \beta y)(\operatorname{sgn}(\alpha)\, |\alpha|^{q-1}, \operatorname{sgn}(\beta)\, |\beta|^{q-1})\|_p \\ &= \|(\alpha, \beta)\|_p \, |\operatorname{sgn}(\beta)\, |\beta|^{q-1} x - \operatorname{sgn}(\alpha)\, |\alpha|^{q-1} y| \\ &\leqslant \|(\alpha, \beta)\|_p \, \|(|\alpha|^{q-1}, |\beta|^{q-1})\|_q \, \|(x, y)\|_p \end{aligned}$$

and the equality is obtained for $x = \operatorname{sgn}(\beta)\, |\beta|^{(q-1)^2}$ and $y = -\operatorname{sgn}(\alpha)\, |\alpha|^{(q-1)^2}$. ∎

We now consider a sequence of Banach spaces $(S_n, \|\cdot\|_n)$ and a real number p, $1 < p < \infty$. We denote by $P_p(S_n)$ the set of all sequences (x_n) such that $x_n \in S_n$ and $\|(x_n)\| := (\sum_{n=1}^{\infty} \|x_n\|_n^p)^{1/p} < \infty$. In [1, p. 35] it is proved that $(P_p(S_n), \|\cdot\|)$ is a Banach space and it is strictly convex (reflexive) iff, for each n, S_n is strictly convex (reflexive). Moreover, if every S_n is reflexive the dual space $(P_p(S_n))^*$ is isometric with $P_q(S_n^*)$, where q is the conjugate of p.

Theorem 2.2. *Let p_n be a sequence of real numbers with $1 < p_n < \infty$. If $p_n \to \infty$ or $p_n \to 1$ then there is a Chebyshev subspace S of $P_p(l_{p_n}(2))$, such that P_S is linear and $\|P_S\| = 2$.*

Proof. Let $(S_n, \|\cdot\|_{p_n})$ be a sequence of subspace of $\mathbb{R}^2$ and $S := P_p(S_n)$. It is easy to show that $P_S((x_n)) = (P_{S_n}(x_n))$, where P_{S_n} is the linear metric projection of $\mathbb{R}^2$ onto S_n; hence S is a Chebyshev subspace and P_S is linear.

Let $x \in \mathbb{R}^2$ and $(x_n) \in P_p(l_{p_n}(2))$ be such that $x_k = x$ and $x_n = 0$ for $n \neq k$. Then $\|P_S((x_n))\| = \|P_{S_k}(x)\|_{p_k}$, thus $\|P_S\| \geq \|P_{S_n}\|$ for all n.

Assume that $p_n \to \infty$. Let $0 < \delta < 1$, $R = 1 + \delta^q$, $\alpha = R^{-1/q}$, and $\beta = \delta R^{-1/q}$; then $\|(\alpha, \beta)\|_q = 1$ and $\|(\alpha, \beta)\|_p \|(|\alpha|^{q-1}, |\beta|^{q-1})\|_q = R^{-1}(1+\delta^p)^{1/p}(1+\delta^{q(q-1)})^{1/q}$. The last expression tends to $2/(1+\delta)$ as $p \to \infty$. Putting δ_n instead of δ in the last expression, with $\delta_n \downarrow 0$, it follows from Lemma 2.1 that there exists a subsequence n_k and numbers α_k, β_k such that $\|P_{S_{n_k}}\|_{p_{n_k}} \geq [2/(1+\delta_k)] - 1/k$, where $S_{n_k} = \{(x, y) : \alpha_k x + \beta_k y = 0\}$. Finally, we define $S = \{(x_n) \in P_p(l_{p_n}(2))\colon x_{n_k} \in S_{n_k}$ and $x_j = 0$, otherwise$\}$. As $\|P_S\| \geq \|P_{S_{n_k}}\|$, we obtain that $\|P_S\| = 2$.

The case $p_n \to 1$ follows analogously, setting $\delta_n \uparrow 1$. ∎

Remark. (1) As $l_{p_n}(2)$ is strictly convex, reflexive space for each n, we have that $P_p(l_{p_n}(2))$ is a strictly convex, reflexive space. Moreover, since $(P_p(l_{p_n}(2)))^* = P_q(l_{q_n}(2))$, where q_n is the conjugate of p_n and $P_q(l_{q_n}(2))$ is strictly convex, it follows that $P_p(l_{p_n}(2))$ is a smooth space [3, p. 106].

(2) As $P_p(l_{p_n})$ is a LUR space [5, Theorem 1.2] and S is a Chebyshev subspace, S is an approximatively compact set. Thus the condition codim $M < \infty$ cannot be removed in Theorem 1.2.

Acknowledgment

We thank the referees for their helpful remarks.

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